

Some scalar curvature warped product splitting theorems

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Abstract

We present a scalar curvature splitting result patterned to some extent after a Ricci curvature splitting result of Croke and Kleiner. The proof is an application of results on marginally outer trapped surfaces. Using a local version of this result (and a variation thereof), we obtain a splitting result for manifolds with boundary that admit a solution to an Obata type equation. This result is relevant to recent work of Lan-Hsuan Huang and the second author concerning aspects of asymptotically locally hyperbolic manifolds.

1 Introduction

In [8], Croke and Kleiner proved the following variation of the Cheeger-Gromoll splitting theorem (see also [15, Theorem C(3)]).

Theorem 1.1. *Let (M, g) be a complete, noncompact n -dimensional ($n \geq 3$) Riemannian manifold with compact boundary N . Assume:*

1. *M has Ricci curvature $\text{Ric}(X, X) \geq -\varepsilon(n - 1)$, for all unit vectors X , where $\varepsilon = 0$ or 1.*
2. *N has mean curvature $H_N \leq -\varepsilon(n - 1)$ (where the mean curvature is defined as the divergence of the inward pointing unit normal).*

Then (M, g) is isometric to $[0, \infty) \times N$, with (warped) product metric $dt^2 + e^{-2\varepsilon t}h$, where h , the induced metric on N , has nonnegative Ricci curvature.¹

¹Note that there is a typo (minus sign missing) in Theorems 1 and 2 in [8].

This theorem is essentially two theorems in one. In the case $\varepsilon = 0$, the conclusion is that M splits as a product, while in the case $\varepsilon = 1$, M splits as a warped product. Their proof uses, by now, very well known techniques in comparison geometry. Note that there are no variational assumptions on N (such as a least area assumption in the case $\varepsilon = 0$).

One of the aims of the present paper is to obtain a kind of scalar curvature version of Theorem 1.1. To motivate one of the assumptions, we introduce the following terminology. Let (M, g) be a Riemannian manifold with compact boundary N having mean curvature $H_N \leq -\varepsilon(n-1)$.² We say that N is *weakly outermost* if there does not exist a compact hypersurface $\Sigma \subset M \setminus N$ cobordant to N satisfying the *strict* mean curvature inequality, $H_\Sigma < -\varepsilon(n-1)$. Without making use of the theorem itself, we observe that the assumptions of Theorem 1.1 imply that N is weakly outermost. If not, there exists a compact hypersurface Σ cobordant to N with $H_\Sigma < -\varepsilon(n-1)$. Now, by standard arguments, there exists an outward directed normal geodesic $\gamma : [0, \infty) \rightarrow M$, with $\gamma(0) \in \Sigma$, such that each initial segment $\gamma|_{[0,t]}$ minimizes the distance from $\gamma(t)$ to Σ (a Σ -ray, if you will). However, by basic comparison geometry, using, e.g. the Riccati equation for the mean curvature of hypersurfaces under normal geodesic flow, the assumption $H_\Sigma < -\varepsilon(n-1)$, together with the Ricci curvature assumption, implies that there must be a focal point to Σ along γ , beyond which γ cannot be minimizing. We remark, as an aside, that the property of being weakly outermost can be used to give an alternative proof of Theorem 1.1.

In our scalar curvature version, we will assume that the scalar curvature S of (M, g) satisfies, $S \geq -\varepsilon n(n-1)$. We will also adopt the assumption that the boundary N is weakly outermost. That however, is not sufficient. Consider the *spatial Schwarzschild manifold*: $M = \mathbb{R}^n \setminus \{r < (\frac{m}{2})^{\frac{1}{n-2}}\}$, with metric (in isotropic coordinates),

$$g = \left(1 + \frac{m}{2r^{n-2}}\right)^{\frac{4}{n-2}} g_E,$$

where g_E is the Euclidean metric and $r = \sqrt{\sum_{i=1}^n x_i^2}$. M has vanishing scalar curvature, $S = 0$, and the boundary $N : r = (\frac{m}{2})^{\frac{1}{n-2}}$ is minimal, $H_N = 0$. Moreover, it follows from the maximum principle for hypersurfaces that N is weakly outermost. The essential ingredient to enforce rigidity is to require that the boundary N not admit a metric of positive scalar curvature (such is the case if N is a torus, for example).

We now state our main result.

Theorem 1.2. *Let (M, g) be a complete, noncompact n -dimensional ($n \geq 3$) Riemannian manifold with compact boundary N . Assume:*

1. M has scalar curvature $S \geq -\varepsilon n(n-1)$, where $\varepsilon = 0$ or 1.
2. N has mean curvature $H_N \leq -\varepsilon(n-1)$.
3. N does not carry a metric of positive scalar curvature and is weakly outermost.

Then (M, g) is isometric to $[0, \infty) \times N$, with (warped) product metric $dt^2 + e^{-2\varepsilon t}h$, where (N, h) is Ricci flat.

²For simplicity we always assume M and N are connected.

The proof makes use of results on marginally outer trapped surfaces. Relevant background is given in Section 2. In Section 3 we present the proof of Theorem 1.2 and some related results. In Section 4, we use a local version of Theorem 1.2 (and a variation thereof) to establish a warped product splitting result for Riemannian manifolds with boundary that admit a nontrivial solution to the equation $\nabla^2 f = fg$. This result plays a role in recent work of Lan-Hsuan Huang and the second author concerning the rigidity of asymptotically locally hyperbolic manifolds of zero mass.

2 Marginally outer trapped surfaces

For the proof of Theorem 1.2, we will make use of the theory of marginally outer trapped surfaces. We begin by recalling some basic definitions and properties. By an initial data set, we mean a triple (M, g, K) , where M is a smooth manifold, g is a Riemannian metric on M and K is a symmetric covariant 2-tensor on M . In general relativity, an initial data set (M, g, K) corresponds to a spacelike hypersurface M with induced metric g and second fundamental form K , embedded in a spacetime (time-oriented Lorentzian manifold) (\bar{M}, \bar{g}) .

Let (M, g, K) be an initial data set. For convenience, we may assume, without loss of generality, that this initial data set is embedded in a spacetime (\bar{M}, \bar{g}) (see e.g. [4, Section 3.2]). While the definition of various quantities is more natural when expressed with respect to an ambient spacetime, all the relevant quantities we introduce depend solely on the initial data set. With respect to the spacetime (\bar{M}, \bar{g}) , the tensor K is given by the following: $K(X, Y) = \bar{g}(\bar{\nabla}_X u, Y)$ for all $X, Y \in T_p M$, where u is the future directed unit normal field to M in \bar{M} .

Let Σ be a closed (compact without boundary) two-sided hypersurface in M . Then Σ admits a smooth unit normal field ν in M , unique up to sign. By convention, refer to such a choice as outward pointing. Then $\ell = u + \nu$ is a future directed outward pointing null normal vector field along Σ . Associated to ℓ is the *null second fundamental form*, χ defined as,

$$\chi : T_p \Sigma \times T_p \Sigma \rightarrow \mathbb{R}, \quad \chi(X, Y) = \bar{g}(\bar{\nabla}_X \ell, Y). \quad (2.1)$$

In terms of the initial data,

$$\chi = K|_{T\Sigma} + A \quad (2.2)$$

where A is the second fundamental form of $\Sigma \subset M$ with respect to the outward unit normal ν . The *null expansion scalar* (or *null mean curvature*) θ of Σ is obtained by tracing χ with respect to the induced metric h on Σ ,

$$\theta = \text{tr}_h \chi = h^{AB} \chi_{AB} = \text{div}_\Sigma \ell. \quad (2.3)$$

Physically, θ measures the divergence of the outgoing light rays emanating from Σ . In terms of the initial data (M, g, K) ,

$$\theta = \text{tr}_h K + H, \quad (2.4)$$

where H is the mean curvature of Σ within M (given by the divergence of ν along Σ).

We say that Σ is outer trapped (resp. weakly outer trapped) if $\theta < 0$ (resp. $\theta \leq 0$) on Σ . If θ vanishes identically along Σ then we say that Σ is a marginally outer trapped surface, or MOTS for short. Such surfaces play an important role in the theory of black holes. Note that in the so-called time-symmetric case, in which $K = 0$, a MOTS is simply a minimal ($H = 0$) surface in M , as follows from (2.4). In this sense, MOTS are a spacetime generalization of minimal surfaces in Riemannian geometry.

2.1 Stability of MOTS.

Unlike minimal surfaces, MOTS in general do not admit a variational characterization. Nevertheless, they admit an important notion of stability which we now describe; cf., [3]. Let Σ be a MOTS in the initial data set (M, g, K) with outward unit normal ν . Consider a normal variation of Σ in M , i.e., a variation $t \rightarrow \Sigma_t$ of $\Sigma = \Sigma_0$ with variation vector field $V = \frac{\partial}{\partial t}|_{t=0} = \phi\nu$, $\phi \in C^\infty(\Sigma)$. Let $\theta(t)$ denote the null expansion of Σ_t with respect to $l_t = u + \nu_t$, where u is the future directed timelike unit normal to M and ν_t is the outer unit normal to Σ_t in M . A computation shows,

$$\left. \frac{\partial \theta}{\partial t} \right|_{t=0} = L(\phi), \quad (2.5)$$

where $L : C^\infty(\Sigma) \rightarrow C^\infty(\Sigma)$ is the operator [3],

$$L(\phi) = -\Delta\phi + 2\langle X, \nabla\phi \rangle + \left(\frac{1}{2}S_\Sigma - (\mu + J(\nu)) - \frac{1}{2}|\chi|^2 + \operatorname{div} X - |X|^2 \right) \phi. \quad (2.6)$$

In the above, Δ , ∇ and div are the Laplacian, gradient and divergence operator, respectively, on Σ , S_Σ is the scalar curvature of Σ , X is the vector field on Σ dual to the one form $X^\flat = K(\nu, \cdot)|_\Sigma$, $\langle \cdot, \cdot \rangle = h$ is the induced metric on Σ , and μ and J are defined in terms of the Einstein tensor $G = \operatorname{Ric}_{\bar{M}} - \frac{1}{2}R_{\bar{M}}\bar{g} : \mu = G(u, u)$, $J = G(u, \cdot)$. When the Einstein equations are assumed to hold, μ and J represent the energy density and linear momentum density along M . As a consequence of the Gauss-Codazzi equations, the quantities μ and J can be expressed solely in terms of initial data,

$$\mu = \frac{1}{2}(S + (\operatorname{tr} K)^2 - |K|^2) \quad \text{and} \quad J = \operatorname{div} K - d(\operatorname{tr} K), \quad (2.7)$$

where S is the scalar curvature on M .

In the time-symmetric ($K = 0$) case, L reduces to the classical stability (or Jacobi) operator of minimal surface theory. As shown in [3], although L is not in general self-adjoint, the eigenvalue $\lambda_1(L)$ of L with the smallest real part, which is referred to as the principle eigenvalue of L , is necessarily real. Moreover there exists an associated eigenfunction ϕ which is strictly positive. The MOTS Σ is then said to be stable if $\lambda_1(L) \geq 0$.

A basic criterion for stability is the following. We say that a MOTS Σ is *weakly outermost* provided there are no outer trapped ($\theta < 0$) surfaces outside of, and cobordant to, Σ . Weakly outermost MOTS are necessarily stable. Indeed, if $\lambda_1(L) < 0$, Equation (2.5), with ϕ a positive eigenfunction ($L(\phi) = \lambda_1(L)\phi$) would then imply that Σ could be deformed outward to an outer trapped surface.

2.2 Rigidity of MOTS

The proof of Theorem 1.2 will be based on two rigidity results for MOTS. The following result was proved by R. Schoen and the author in [12].

Theorem 2.1 (infinitesimal rigidity). *Let (M, g, K) be an initial data set that satisfies the **dominant energy condition** (DEC),*

$$\mu \geq |J| \quad \text{along } M, \quad (2.8)$$

where μ and J are as in (2.7). *If Σ is a stable MOTS in M that does not admit a metric of positive scalar curvature then*

1. Σ is Ricci flat.
2. $\chi = 0$ and $\mu + J(\nu) = 0$ along Σ .

By strengthening the stability assumption, namely by requiring the MOTS Σ to be weakly outermost, as defined at the end of Section 2.1, we obtain additional rigidity. The following was proved in [11].

Theorem 2.2. *Let (M, g, K) , be an initial data set satisfying the DEC. Suppose Σ is a weakly outermost MOTS in M that does not admit a metric of positive scalar curvature. Then there exists an outer neighborhood $U \approx [0, \delta) \times \Sigma$ of Σ in M such that each slice $\Sigma_t = \{t\} \times \Sigma$, $t \in [0, \delta)$ is a MOTS.*

Remark. It follows again from the discussion at the end of Section 2.1 that, in the theorem above, each MOTS Σ_t is stable, as otherwise Σ would not be weakly outermost.

3 Proof of Theorem 1.2 and related results

Theorem 1.2, in fact, follows easily from a corresponding local splitting result. Let (M, g) be a Riemannian manifold with compact boundary N having mean curvature $H_N \leq -\varepsilon(n-1)$. Then we say that N is *locally weakly outermost* provided there is a neighborhood U of N such that N is weakly outermost in $(U, g|_U)$.

Theorem 3.1. *Let (M, g) be an n -dimensional ($n \geq 3$) Riemannian manifold with compact boundary N . Assume:*

1. M has scalar curvature $S \geq -\varepsilon n(n-1)$, where $\varepsilon = 0$ or 1 .
2. N has mean curvature $H_N \leq -\varepsilon(n-1)$.
3. N does not carry a metric of positive scalar curvature and is locally weakly outermost.

Then there exists a neighborhood V of N such that $(V, g|_V)$ is isometric to $[0, \delta) \times N$, with (warped) product metric $dt^2 + e^{-2\varepsilon t}h$, where (N, h) is Ricci flat.

Proof. The proof of Theorem 3.1 consists primarily of applying the MOTS rigidity results in Section 2.2 to the initial data set (M, g, K) , where (M, g) satisfies the assumptions of the theorem and $K = \varepsilon g$. We are assuming that there is a neighborhood U of N such that N is weakly outermost in $(U, g|_U)$.

We first observe that, with respect to the initial data set $(M, g, K = \varepsilon g)$, the DEC (2.8) holds. Inserting $K = \varepsilon g$ into the expression for μ in (2.7) leads to

$$\mu = \frac{1}{2}(S + \varepsilon^2 n(n-1)) = \frac{1}{2}(S + \varepsilon n(n-1)). \quad (3.9)$$

Hence, by property 1 of Theorem 3.1, $\mu \geq 0$. Further, $K = \varepsilon g$ implies $J = 0$, so that $\mu + |J| \geq 0$, and the DEC is satisfied.

Next, let's consider the null expansion of N . Equation (2.4) implies that N has null expansion,

$$\theta = \varepsilon(n-1) + H_N. \quad (3.10)$$

Hence by property 2 of Theorem 1.2, $\theta \leq 0$, i.e. N is weakly outer trapped. In fact one must have $\theta \equiv 0$. Otherwise, it follows from [4, Lemma 5.2], that, by a small perturbation of N , there would exist a strictly outer trapped ($\theta < 0$) compact hypersurface $N' \subset U$ outside of, and cobordant to N , thereby contradicting the assumption that N is weakly outermost in U .

Hence, N is a weakly outermost MOTS in U . So, by Theorem 2.2, we can introduce coordinates (t, x^i) on a neighborhood $V = [0, \delta) \times N$ of N in U , so that g in these coordinates may be written as,

$$g = \psi^2 dt^2 + h_{ij} dx^i dx^j, \quad (3.11)$$

where $\psi = \psi(t, x^i)$ is positive, $h_t = h_{ij}(t, x^i) dx^i dx^j$ is the induced metric on $N_t = \{t\} \times N$, and N_t is a MOTS, $\theta(t) = 0$.

A computation similar to that leading to (2.5) (but where for the moment we do not assume $\theta = \theta(t)$ vanishes) leads to the following 'evolution equation' for $\theta = \theta(t, x^i)$ ([5, 9]),

$$\frac{\partial \theta}{\partial t} = -\Delta \psi + 2\langle X_t, \nabla \psi \rangle + \left(Q_t - \frac{1}{2}\theta^2 + \theta \operatorname{tr} K + \operatorname{div} X_t - |X_t|^2 \right) \phi, \quad (3.12)$$

$$Q_t = \frac{1}{2}S_{N_t} - (\mu + J(\nu)) - \frac{1}{2}|\chi_t|^2, \quad (3.13)$$

where it is understood that, for each t , the above terms live on Σ_t , e.g., $\Delta = \Delta_t$ is the Laplacian on N_t , $\langle \cdot, \cdot \rangle = h_t$, $X_t^\flat = K(\nu_t, \cdot)|_{N_t}$, etc.

Note from the form of K , $X_t = 0$. Setting $\theta = 0$ and $X_t = 0$ in (3.12), and using (3.13), we obtain,

$$\Delta \psi + ((\mu + J(\nu)) + \frac{1}{2}|\chi_t|^2 - \frac{1}{2}S_{N_t})\psi = 0. \quad (3.14)$$

By the remark following Theorem 2.2, each N_t is a stable MOTS. Hence, by Theorem 2.1,

$$N_t \text{ is Ricci flat, } \chi_t = 0, \text{ and } \mu + J(\nu) = 0. \quad (3.15)$$

Equation (3.14) then becomes,

$$\Delta\psi = 0,$$

and, hence, ψ is constant along each N_t , $\psi = \psi(t)$. By a simple change of variable, we thus may assume $\psi = 1$, and so (3.11) becomes,

$$g = dt^2 + h_{ij}dx^i dx^j. \quad (3.16)$$

From (2.2), $\chi_t = K|_{TN_t} + A_t = \varepsilon h_t + A_t$ where A_t is the second fundamental form of N_t . Then, from the second equation in (3.15), $A_t = -\varepsilon h_t$, which becomes, in the coordinate expression (3.16), $\frac{\partial h_{ij}}{\partial t} = -2\varepsilon h_{ij}$. Integrating gives, $h_{ij}(t, x) = e^{-2\varepsilon t} h_{ij}(0, x)$. Thus, up to isometry, we have $V = [0, \delta) \times N$, $g|_V = dt^2 + e^{-2\varepsilon t} h$. \square

Proof of Theorem 1.2. By Theorem 3.1, there exists a neighborhood V of N such that $(V, g|_V)$ is isometric to $([0, \delta) \times N, dt^2 + e^{-2\varepsilon t} h)$. By the completeness assumption, it is clear that this warped product structure extends to $t = \delta$. From the fact that N is weakly outermost, it follows that $N_\delta = \{\delta\} \times N$ is weakly outermost. Theorem 3.1 then implies that the warped product structure extends beyond $t = \delta$. By a continuation argument, it follows that the warped product structure exists for all $t \in [0, \infty)$. \square

Although the motivation is now somewhat different, we note that, by using the the initial data set $(M, g, K = -\varepsilon g)$, one can show by very similar arguments the following variation of Theorem 3.1.

Theorem 3.2. *Let (M, g) be an n -dimensional ($n \geq 3$) Riemannian manifold with compact boundary N . Assume:*

1. M has scalar curvature $S \geq -\varepsilon n(n-1)$, where $\varepsilon = 0$ or 1 .
2. N has mean curvature $H_N \leq \varepsilon(n-1)$.
3. N does not carry a metric of positive scalar curvature and is locally weakly outermost.

Then there exists a neighborhood V of N such that $(V, g|_V)$ is isometric to $[0, \delta) \times N$, with (warped) product metric $dt^2 + e^{2\varepsilon t} h$, where (N, h) is Ricci flat.

In lieu of the weakly outermost assumption, in the case of mean curvature equality, $H_N = \varepsilon(n-1)$, similar splitting results have been obtained, under the assumption that N is area minimizing ($\varepsilon = 0$, [7, 6, 10]) or that N minimizes the ‘brane action’ ($\varepsilon = 1$, [1]).

Remark. Theorem 3.2 has the following consequence. Let (M, g) be an n -dimensional, $3 \leq n \leq 7$, asymptotically flat manifold with compact minimal boundary N , and with nonnegative scalar curvature, $S \geq 0$. Suppose, further, that N is an outermost minimal surface, i.e. suppose that there are no minimal surfaces in $M \setminus N$ homologous to N . Then N necessarily carries a metric of positive scalar curvature. For, suppose not. To apply Theorem 3.2 in the case $\varepsilon = 0$, it is sufficient to show that N is locally weakly outermost. If that were not the case, there would exist a compact hypersurface N_1 cobordant to N with mean curvature $H_1 < 0$. On the other hand sufficient far out on the asymptotically flat end

there exists a compact hypersurface N_2 cobordant to N_1 with mean curvature $H_2 > 0$. N_1 and N_2 bound a region W . Basic existence results for minimal surfaces (or for MOTS [2]), guarantee the existence of a minimal surface in W homologous to N , contrary to assumption. Hence N is weakly outermost. Theorem 3.2 then implies that (M, g) locally splits near N , contrary to N being an outermost minimal surface. The same consequence holds for an n dimensional, $3 \leq n \leq 7$, asymptotically hyperbolic manifold (M, g) with compact boundary N of constant mean curvature $n-1$, and with scalar curvature $S \geq -n(n-1)$ in the following sense: Consider the initial data set $(M, g, -g)$, with (M, g) as just described, and suppose N is an outermost MOTS. Then N necessarily carries a metric of positive scalar curvature.

Similar to Theorem 3.1, Theorem 3.2 implies the following global result.

Theorem 3.3. *Let (M, g) be a complete, noncompact n -dimensional ($n \geq 3$) Riemannian manifold with compact boundary N . Assume:*

1. M has scalar curvature $S \geq -\varepsilon n(n-1)$, where $\varepsilon = 0$ or 1 .
2. N has mean curvature $H_N \leq \varepsilon(n-1)$.
3. N does not carry a metric of positive scalar curvature and is weakly outermost.

Then (M, g) is isometric to $[0, \infty) \times N$, with (warped) product metric $dt^2 + e^{2\varepsilon t}h$, where (N, h) is Ricci flat.

4 Warped product splitting and Obata's equation

The main aim of this section is to prove the following.

Theorem 4.1. *Let (M, g) be an n -dimensional ($n \geq 3$) complete, noncompact Riemannian manifold with compact boundary N . Let $h = g|_N$. Suppose that*

1. $S \geq -n(n-1)$ in a neighborhood of N .
2. N has mean curvature $H_N \leq \delta(n-1)$, where $\delta = 1$ or -1 .
3. N does not carry a metric of positive scalar curvature and is locally weakly outermost.
4. There exists a nonzero function f satisfying $\nabla^2 f = fg$.

Then (M, g) is isometric to $[0, \infty) \times N$, with warped product metric $dt^2 + e^{2\delta t}h$ where (N, h) is Ricci flat. In particular, if (N, h) is flat, then (M, g) is of constant sectional curvature -1 .

Note that the resulting warped product corresponds to an unbounded portion of the hyperbolic cusp: it contains either an expanding end when $\delta = 1$ or a shrinking end when $\delta = -1$.

As remarked in the Introduction, this theorem plays a role in recent work of Lan-Hsuan Huang and the second author [13] concerning the rigidity of asymptotically locally hyperbolic manifolds of zero mass. Obata's equation in the form $\nabla^2 f = fg$ has been studied previously

in the literature; see e.g. [14, 16]. In addition to Theorems 3.1 and 3.2, the proof of Theorem 4.1 will make use of the following result, which extends to manifolds with boundary certain results in [14].

Proposition 4.2. *Let (M^n, g) be a complete connected Riemannian manifold with compact connected boundary N ($n \geq 3$). Suppose there exists a nonzero function f that satisfies*

$$\nabla^2 f = fg, \quad (4.17)$$

and N is a regular hypersurface $f^{-1}(a)$ for $a \in \mathbb{R}$. Then the following hold:

1. If M is compact, then (M, g) is isometric to a hyperbolic cap $[0, R] \times \mathbb{S}^{n-1}$ equipped with the metric

$$dt^2 + (\sinh t)^2 g_{\mathbb{S}^{n-1}}$$

where $g_{\mathbb{S}^{n-1}}$ is the standard unit sphere metric and $R = d_g(p, N)$ for $p \in M \setminus N$ which is a critical point of f .

2. If M is noncompact, then (M, g) is isometric to a manifold $[0, \infty) \times N$ with (warped) product metric of the form

$$dt^2 + \xi(t)^2 g|_N,$$

where $\xi : [0, \infty) \rightarrow \mathbb{R}$ is the solution to the following ODE

$$\begin{cases} \xi'' - \xi = 0 \text{ on } [0, \infty), \\ \xi(0) = 1 \text{ and } \xi'(0) = \frac{a}{|\nabla f|_N}. \end{cases} \quad (4.18)$$

(We note, as follows from (4.17), that $|\nabla f|_N$ is constant.)

Proof of Proposition 4.2. First we claim that f has a critical point on the interior of M if and only if M is compact (with boundary).

Suppose that f has a critical point p in M . Consider a unit speed geodesic $\gamma : [0, \infty) \rightarrow M$ emanating from p . It follows that

$$\frac{d^2}{dr^2} f(\gamma(r)) - f(\gamma(r)) = 0$$

thus $f(\gamma(r)) = c(e^r + e^{-r})$ and $\frac{d}{dr} f(\gamma(r)) = c(e^r - e^{-r})$, where $c \neq 0$ (as otherwise f would vanish identically). Observe that f depends only on the geodesic distance from the point p , which implies that γ' is parallel to ∇f . Moreover, there cannot be any other critical point of f . Let $R = \text{dist}(p, N)$. Then it follows that $N = \exp_p(S_R)$, and from this that $\exp_p : \overline{B_R} \rightarrow M$ is bijective. By continuity of the exponential map, this implies that M must be compact.

Suppose, conversely, M is compact. For contradiction, suppose also that f has no critical points. Without loss of generality, we may assume that ∇f points inward on N . Let $\nu = \nabla f / |\nabla f|$, and consider the integral curve γ of ν emanating from a point $p \in N$, i.e.,

$\gamma(0) = p$. It is straightforward that γ is a geodesic parametrized by arc length, and we also have

$$f \circ \gamma(t) = c_1 e^t + c_2 e^{-t}$$

as we observed before. Since f has no critical point, γ can be extended to $[0, \infty)$, which implies that γ is an injective infinite length geodesic. This contradicts the condition that M is compact, hence f must have a critical point on the interior of M .

We now show the first case of the proposition: assume that M is compact. From the previous argument, there is a critical point p such that $\exp_p : \overline{B_R} \rightarrow M$ is bijective where $R = d_g(p, N)$. Now we show that it is a diffeomorphism. Let J be a Jacobi field along γ such that $J(0) = 0$ and $|J'(0)| = 1$ and $g(J, \gamma') = 0$. Then we have for $r > 0$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \Big|_{t=r} g(J, J) &= g(J, \nabla_{\gamma'} J) \Big|_{t=r} \\ &= A(J, J) \Big|_{t=r} = \frac{f}{|\nabla f|} g(J, J) \Big|_{t=r}, \end{aligned}$$

where $A = \nabla^2 f / |\nabla f|$ is the second fundamental form of the geodesic spheres. Thus for $r > r_0 > 0$,

$$|J|^2(r) = \left(\frac{e^r - e^{-r}}{e^{r_0} - e^{-r_0}} \right)^2 |J|^2(r_0) \neq 0$$

where r_0 is sufficiently small that $|J(r_0)| \neq 0$. This implies that there is no conjugate point from p thus $\exp_p : \overline{B_R} \rightarrow M$ is a diffeomorphism. Furthermore, by using geodesic polar coordinates, we can write the metric g on M diffeomorphic to $[0, R] \times \mathbb{S}^{n-1}$ as

$$g = dt^2 + (\sinh t)^2 g_{\mathbb{S}^{n-1}}.$$

We turn to the second case: assume that M is noncompact. Let $h = g|_N$. We will construct an isometry between (M, g) and the manifold

$$([0, \infty) \times N, dt^2 + \xi^2 h)$$

where ξ is given in (4.18). Without loss of generality, we may assume that ∇f points inward on N .

Let φ be the flow generated by $\nu = \nabla f / |\nabla f|$, and define the map $\psi : [0, \infty) \times N \rightarrow M$ by

$$\psi(t, \bar{p}) = \varphi_t(\bar{p})$$

for $\bar{p} \in N$ and $t \in [0, \infty)$. Since f has no critical points, it is clear that ψ is a diffeomorphism.

As we observed before, we have the general solution

$$f \circ \psi(\bar{p}, t) = c_1 e^t + c_2 e^{-t}, \quad \bar{p} \in N, t \in [0, \infty),$$

where the constants c_1 and c_2 are determined by the conditions on f at N ; specifically, $c_1 + c_2 = a$ and $c_1 - c_2 = |\nabla f|_N$. In terms of these constants, the solution to the IVP (4.18) is given by, $\xi(t) = \frac{c_1 e^t - c_2 e^{-t}}{c_1 - c_2}$.

Now we prove that ψ is the desired isometry from $([0, \infty) \times N, dt^2 + \xi(t)^2 h)$ onto (M, g) . Using ψ as a coordinate chart, we can write the metric

$$g = dt^2 + g_{ij}(t, \bar{p}) dx^i dx^j$$

where $\{x^i\}_{i=1}^{n-1}$ are local coordinates near \bar{p} on N and $g_{ij}(t, \bar{p}) = g_{(t, \bar{p})}(\partial_i, \partial_j)$ for $t \in [0, \infty)$. Then we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \Big|_{t=\tau} g_{ij}(t, \bar{p}) &= g_{(\tau, \bar{p})}(\partial_i, \nabla_\nu \partial_j) = A_{(\tau, \bar{p})}(\partial_i, \partial_j) \\ &= \frac{f(\tau)}{|\nabla f|(\tau)} g_{ij}(\tau, \bar{p}) = \frac{c_1 e^\tau + c_2 e^{-\tau}}{c_1 e^\tau - c_2 e^{-\tau}} g_{ij}(\tau, \bar{p}) \end{aligned}$$

where $A_{(\tau, \bar{p})}$ is the second fundamental form of the hypersurface $f^{-1}(f \circ \gamma(\tau))$. Thus we obtain

$$g_{ij}(\tau, \bar{p}) = \xi(\tau)^2 g_{ij}(0, \bar{p}) = \xi(\tau)^2 h_{ij}(\bar{p}),$$

and by varying $(\tau, \bar{p}) \in M$ it proves that ψ is the desired isometry. \square

Proof of Theorem 4.1. We will only prove $\delta = 1$ since the proof of the other case is almost identical.

By Theorem 3.2 (with $\varepsilon = 1$), we have the local splitting near N , that is, there exists a neighborhood U of N such that U is isometric to $[0, b) \times N$ for some $b > 0$ with the metric $dt^2 + e^{2t} h$. To use Proposition 4.2, we show that N is the level set $f^{-1}(a)$ for some $a \in \mathbb{R}$.

Let $\{x^i\}_{i=1}^{n-1}$ be local coordinates on N . This gives rise to local coordinates $\{t = x_0, x_1, \dots, x_{n-1}\}$ on $[0, b) \times N$ in the obvious manner. Then, by direct computation, we have

$$\begin{aligned} 0 &= \nabla_{\partial_t} \nabla_{\partial_i} f = \partial_t \partial_i f - \sum_{k=0}^{n-1} \Gamma_{ti}^k \partial_k f \\ &= \partial_t \partial_i f - \partial_i f, \end{aligned} \tag{4.19}$$

$$f = \nabla_{\partial_t} \nabla_{\partial_t} f = \partial_t^2 f, \tag{4.20}$$

$$e^{2t} f h_{ij} = \nabla_{\partial_i} \nabla_{\partial_j} f = \partial_i \partial_j f + e^{2t} h_{ij} \partial_t f - \sum_{l=1}^{n-1} \bar{\Gamma}_{ij}^l \partial_l f. \tag{4.21}$$

where $\bar{\Gamma}$ is the Christoffel symbol with respect to h . Denote $f = f(p, t)$ on U for $p \in N$ and $t \in [0, b)$. Then from the above computations we have

$$\partial_t^2 f - f = 0 \Rightarrow f(p, t) = c_1(p) e^t + c_2(p) e^{-t}, \tag{4.22}$$

$$\partial_t(\partial_i f) - \partial_i f = 0 \Rightarrow \partial_i f(p, t) = c_3(p) e^t, \tag{4.23}$$

It follows from (4.22) and (4.23) that $c_2(p)$ is constant on N , hence we can write

$$f(p, t) = c_1(p) e^t + c_2 e^{-t}, \text{ and } \partial_t f - f = -2c_2 e^{-t}.$$

Now we shall show that $c_1(p)$ is constant on N . By (4.21), we have

$$\begin{aligned}
\partial_i \partial_j f + e^{2t} h_{ij} (\partial_i f - f) - \sum_{l=1}^{n-1} \bar{\Gamma}_{ij}^l \partial_l f &= 0 \\
\Rightarrow e^t \left(\partial_i \partial_j (c_1(p)) - 2c_2 h_{ij} - \sum_{l=1}^{n-1} \bar{\Gamma}_{ij}^l \partial_l c_1(p) \right) &= 0 \\
\Rightarrow \nabla_{\partial_i}^N \nabla_{\partial_j}^N c_1 = 2c_2 h_{ij} \Rightarrow \Delta_N c_1 = 2(n-1)c_2. & \quad (4.24)
\end{aligned}$$

Since N is compact without boundary, we have

$$0 = \int_N \Delta_N c_1 = 2(n-1)c_2 |N|$$

where $|N|$ is the area of N . This implies that $c_2 = 0$, and hence $c_1(p)$ is harmonic on N . Therefore $c_1(p)$ is constant on N so $N = f^{-1}(c_1)$.

By Proposition 4.2, (M, g) is isometric to $[0, \infty) \times N$ with the metric $dt^2 + \xi(t)^2 h$. In particular, one can see that the warping factor is $\xi(t) = e^t$. \square

ACKNOWLEDGEMENTS. GJG was partially supported by NSF grant DMS-1710808. HCJ was partially supported by NSF Grant DMS-1452477. The authors would like to thank Lan-Hsuan Huang for her interest in this work and for many helpful comments. HCJ is especially grateful to Professor Huang for her constant encouragement and guidance.

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